



Barker College

2010 TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

PM THURSDAY 5TH AUGUST

- MRB*
- GDH
- BHC
- WMD

40 copies

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages of your answer sheets
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper

Total marks – 120

- Attempt Questions 1–8
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

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Total marks – 120

Attempt Questions 1– 8

Answer each question on a **SEPARATE** sheet of paper

		Marks
Question 1	(15 marks) [START A NEW PAGE]	
(a)	Find $\int x^2 \log_e x \, dx$	2
(b)	Find $\int \frac{\cos^2 x}{\sin^2 x} \, dx$	2
(c)	Find $\int \frac{\cos^3 x}{\sin^2 x} \, dx$	3
(d)	Using the substitution of $t = \tan \frac{\theta}{2}$ or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3 - \cos \theta}$	4
(e)	(i) Find the remainder when $x^2 + 6$ is divided by $x^2 + x - 6$.	1
	(ii) Hence, find $\int \frac{x^2 + 6}{x^2 + x - 6} \, dx$	3

End of Question 1

Question 2 (15 marks) **[START A NEW PAGE]**

(a) Let $z = 18 + 4i$ and $w = 3 - i$

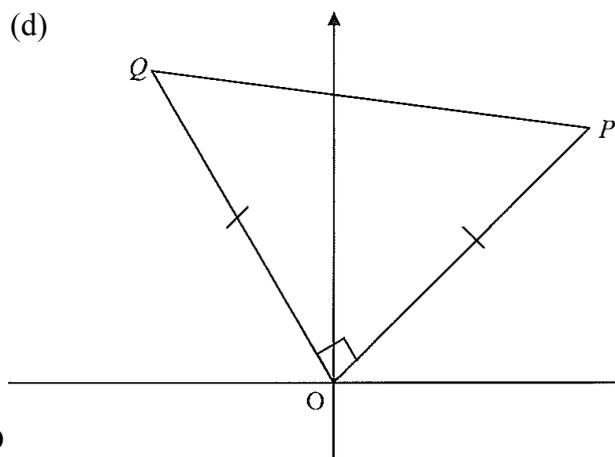
(i) Express $\frac{z}{w}$ in the form $a + ib$, where a and b are real numbers 2

(ii) Hence or otherwise find $\left| \frac{z}{w} \right|$ and $\arg\left(\frac{z}{w}\right)$ 2

(b) Sketch the region in the complex plane where the inequalities

$$|z - 1 + 2i| \leq 3 \quad \text{and} \quad \left| \arg(z - 1 + 2i) \right| \leq \frac{\pi}{3} \quad \text{both hold.} \quad \text{3}$$

(c) Evaluate: $\left(\frac{1+i}{\sqrt{2}}\right)^{2010}$ 2



The points P and Q in the complex plane correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles and $\angle POQ$ is a right angle.

Show that $z^2 + w^2 = 0$ 1

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(e) (i) By solving the equation $z^3 - 1 = 0$, find the three cube roots of 1. 2

(ii) Let λ be a cube root of 1 where λ is not real. Show that $\lambda^2 = -(\lambda + 1)$ 2

(iii) Hence simplify $(1 + \lambda)^6$ 1

End of Question 2

Question 3 (15 marks) **[START A NEW PAGE]**

(a) By considering the quadratic equation $x^2 - xy + 1 = 0$ in terms of x ,
Show $|y| \geq 2$ 2

(b) Let $f(x) = x + \frac{1}{x}$

(i) By considering your result in (a) or otherwise, sketch a half page
graph of $y = f(x)$. 2

Provide separate half page sketches of the following functions.

(ii) $y = |f(x)|$ 1

(iii) $y = \frac{1}{f(x)}$ 2

(iv) $y = \sqrt{f(x)}$ 2

(v) $y = e^{f(x)}$ 2

(c) (i) The number of original thoughts (T) in a small community is given by
the equation $\frac{dT}{dt} = k(10000 - T)$.
Show that $T = 10000 - Ae^{-kt}$ where A, k are constants, is a solution of
this equation. 1

(ii) Initially there were 2500 original thoughts and then 10 years later this had
risen to 5000. When will the number of original thoughts be 7500? 3

End of Question 3

Question 4 (15 marks) **[START A NEW PAGE]**

- (a) (i) Suppose that k is a double root of the polynomial equation $f(x) = 0$. Show that $f'(k) = 0$. 2
- (ii) The polynomial $P(x) = ax^7 + bx^6 + 1$ is divisible by $(x - 1)^2$. Find a and b . 2
- (iii) Let $E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$. Prove $E(x) = 0$ has no double root. 2
- (b) The roots of $x^3 + 7x^2 + 13 = 0$ are α, β, γ ,
- (i) Find the polynomial equation whose roots are $\alpha^2, \beta^2, \gamma^2$. 2
- (ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ 1
- (c) (i) By first finding the turning points of $p(x) = x^3 - x^2 - 5x - 1$, sketch the graph of $y = p(x)$. 2
- (ii) Without attempting to solve the equation, explain why $p(x) = 0$ has three distinct real roots, two of which are negative. 1
- (iii) Starting with an approximation of $x = 0$, use one application of Newton's method to estimate a root of the equation $p(x) = 0$. 2
- (iv) Using Newton's method, what initial approximation would you use to estimate the positive root of $p(x) = 0$? State briefly your reasons for this choice. 1

End of Question 4

Question 5 (15 marks) **[START A NEW PAGE]**

- (a) The ellipse Σ has the cartesian equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$.
- (i) Sketch Σ clearly labelling the foci, S and S' and directrices 3
- (ii) P is an arbitrary point on Σ . By using the focus directrix definition, prove that the sum of the distances from P to the two foci, S and S' is independent of the position of P . 2
- (iii) The point $Q(x_1, y_1)$ is located on the ellipse Σ where $x_1 > 0$ and $y_1 > 0$.
The tangent at Q meets the directrix at $M(x_2, y_2)$ where $x_2 > 0$.
Show that QM subtends a right angle at one of the foci. 4
- (b) The line $L : ax + by = 1$ meets the rectangular hyperbola $H : xy = c^2$ in two distinct points, $A(x_1, y_1)$ and $B(x_2, y_2)$.
 $C(x_0, y_0)$ is the mid-point of AB .
- (i) Find a quadratic equation whose roots are x_1 and x_2 . 2
- (ii) Show that the equation of L may be written $\frac{x}{x_0} + \frac{y}{y_0} = 2$. 2
- (iii) A line M (other than an asymptote) passes through the centre of a rectangular hyperbola. This line bisects all chords parallel to a certain direction. What is the direction of these chords in terms of the direction of M ? 2

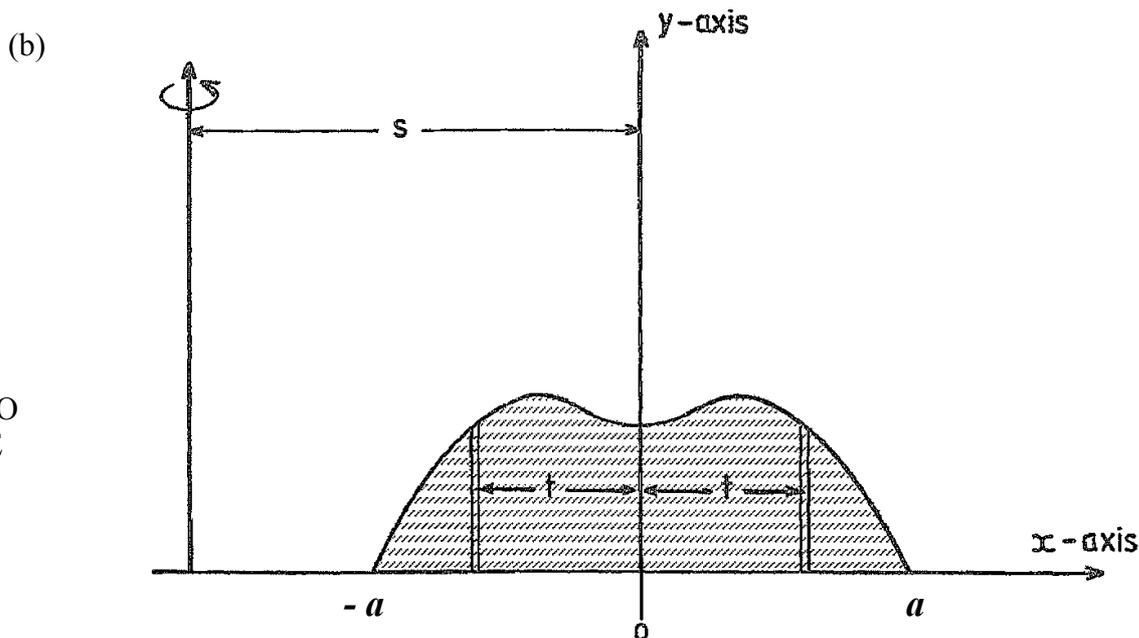
End of Question 5

Question 6 (15 marks) **[START A NEW PAGE]**

(a)
$$I_n = \int_0^1 x^n e^x dx$$

(i) Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \dots$ 2

(ii) Hence evaluate I_2 1



The diagram shows the shaded area A between the smooth curve $y = f(x)$, $-a \leq x \leq a$, and the x -axis. (Note that $f(x) \geq 0$ for $-a \leq x \leq a$ and $f(-a) = f(a) = 0$).

The area A is rotated about the line $x = -s$ (where $s > a$) to generate the volume V .

This volume is to be found by slicing A into thin vertical strips, rotating these to obtain cylindrical shells, and adding the shells. Two typical strips of width δt whose centre lines are distance t from the y -axis are shown.

(i) Show that the indicated strips generate shells of approximate volume $2\pi f(-t)(s - t)\delta t$ and $2\pi f(t)(s + t)\delta t$ respectively. 1

(ii) Assuming that $f(x)$ is an even function, show that $V = 2\pi s A$. 2

Question 6 continues on page 9

Question 6 (continued)

Assuming the result of part (ii), solve the following problems.

- (iii) A doughnut shape is formed by rotating a circular disc of radius r about an axis in its own plane at a distance of s ($s > r$) from the centre of the disc. Find the volume of the doughnut. **1**
- (iv) The shape of a certain party jelly can be represented by rotating the area between the curve $y = \sin x$, $0 \leq x \leq \pi$, and the x -axis about the line $x = -\pi/4$. Find the volume generated. **2**
- (c) (i) By considering $\int_0^a \sqrt{a^2 - x^2} dx$ show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab units². **2**

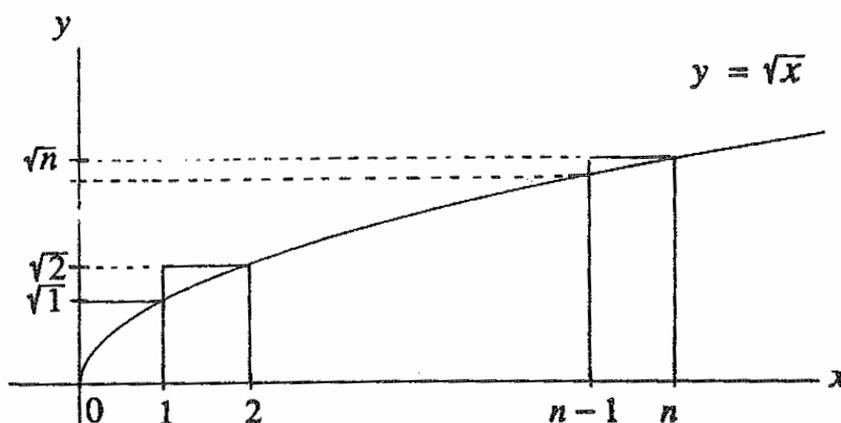
A flag pole of height d metres is constructed on an elliptical base where the length of the major axis is twice the length of the minor axis. The pole tapers to a circular cross section whose radius is half that of the semi-minor axis. Let the radius of this circle be r metres.

- (ii) Show that the area of the elliptical cross section h metres above the ground is given by $\frac{\pi r^2}{d^2} (3h^2 - 10hd + 8d^2)$ square metres. **2**
- (iii) Calculate the volume of the flag pole in terms of r and d . **2**

End of Question 6

Question 7 (15 marks) [START A NEW PAGE]

(a)



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Consider the graph of the function $y = \sqrt{x}$.

- (i) Show that this curve is increasing for all $x > 0$. 1

- (ii) Explain why $\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \int_0^n \sqrt{x} \, dx$. 1

- (iii) Hence show $\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \frac{2}{3} n \sqrt{n}$. 1

- (iv) By squaring both sides, show that $(4k + 3)\sqrt{k} < (4k + 1)\sqrt{k + 1}$ for $k > 0$. 1

- (v) Use mathematical induction to show that $\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \leq \frac{4n + 3}{6} \sqrt{n}$ for all integers $n \geq 1$. 2

- (vi) Hence, estimate $\sqrt{1} + \sqrt{2} + \dots + \sqrt{10000}$ to the nearest hundred. 1

Question 7 continues on page 11

Question 7 (continued)

(b) Let $q = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$

The complex number $\alpha = q + q^2 + q^4$ is a root of the quadratic equation

$x^2 + ax + b = 0$, where a and b are real.

(i) Prove that $1 + q + q^2 + q^3 + q^4 + q^5 + q^6 = 0$. 2

(ii) The second root of the quadratic equation is β . Express β in terms of positive powers of q . 1

(iii) Find the values of the coefficients a and b . 2

(iv) Deduce:

$$\sin \frac{\pi}{7} - \sin \frac{2\pi}{7} - \sin \frac{3\pi}{7} = \frac{-\sqrt{7}}{2}. \quad 3$$

End of Question 7

Question 8 (15 marks) **[START A NEW PAGE]****Marks**

- (a) The body of mass one kilogram is projected vertically upwards from the ground at a speed of 20 metres per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the magnitude of the particle's velocity at that time.

In the following questions take the acceleration due to gravity to be 10 metres per second per second.

While the body is travelling upwards the equation of motion is: $\ddot{x} = -\left(10 + \frac{1}{40}v^2\right)$

- (i) Taking $\dot{x} = v \frac{dv}{dx}$, calculate the greatest height reached by the particle. **3**

- (ii) Taking $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach this greatest height. **2**

Having reached its greatest height the particle falls to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$.

- (iii) Write down the equation of motion of the particle as it falls. **1**

- (iv) Find the speed of the particle when it returns to its starting point. **3**

- (b) Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection (upward inclination to the horizontal), α , but with different velocities V_1 and V_2 metres per second where $V_1 < V_2$.

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

- (i) Show that, while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time. **2**

- (ii) Hence, express the horizontal distance from P to the foot of the wall in the terms of h and α . **1**

- (iii) Show that $V_2(\tan \alpha + \tan \beta) = 2V_1 \tan \alpha$, and deduce that, if $\beta = \frac{1}{2}\alpha$, then $V_1 < \frac{3}{4}V_2$. **3**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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① (a) $\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$
 $= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx$
 $= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$

(ii) $\int \frac{x^2+6}{x^2+x-6} \, dx = \int 1 + \frac{12-x}{x^2+x-6} \, dx$

Find A, B such that $\frac{12-x}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$

ie $12-x = A(x-2) + B(x+3)$

If $x = -3$: $-5A = 15 \quad \therefore A = -3$

If $x = 2$: $5B = 10 \quad \therefore B = 2$

$\therefore \int 1 - \frac{3}{x+3} + \frac{2}{x-2} \, dx$

$= x - 3 \ln|x+3| + 2 \ln|x-2| + C$

(b) $\int \frac{\cos^2 x}{\sin^2 x} \, dx = \int \cot^2 x \, dx = \int \operatorname{cosec}^2 x - 1 \, dx$
 $= -\cot x - x + C$

(c) $\int \frac{\cos^2 x \cos x}{\sin^2 x} \, dx = \int \frac{\cos x - \sin^2 x \cos x}{\sin^2 x} \, dx$
 $= \int \frac{\cos x}{\sin^2 x} \, dx - \int \cos x \, dx$
 $= \int \cos x (\sin x)^{-2} \, dx - \sin x + C$
 $= \frac{-1}{\sin x} - \sin x + C$

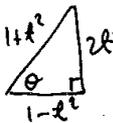
② (a) (i) $\frac{18+4i}{3-i} \times \frac{3+i}{3+i} = \frac{50+30i}{10} = 5+3i$

(ii) $\sqrt[3]{34}$ $\therefore \left| \frac{z}{w} \right| = \sqrt[3]{34}$, $\arg\left(\frac{z}{w}\right) = \tan^{-1} \frac{3}{5}$

(d) $t = \tan \frac{\theta}{2} \quad \frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{\theta}{2} = \frac{1}{2} (1 + \tan^2 \frac{\theta}{2})$
 $= \frac{1+t^2}{2}$

$\therefore 2dt = (1+t^2) d\theta$

$\therefore d\theta = \frac{2dt}{1+t^2}$



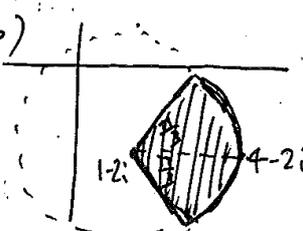
When $\theta = 0, t = 0$
 $\theta = \frac{\pi}{2}, t = \tan \frac{\pi}{4} = 1$

$\therefore \int_0^1 \frac{2dt}{(1+t^2)(3 - \frac{1-t^2}{1+t^2})}$

$= 2 \int_0^1 \frac{dt}{3+3t^2-1+t^2}$

$= 2 \int_0^1 \frac{dt}{2+4t^2} = \int_0^1 \frac{dt}{1+2t^2} = \frac{1}{2} \int_0^1 \frac{dt}{\frac{1}{2}+t^2}$

$= \frac{1}{2} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \left[\tan^{-1} \frac{t}{\frac{1}{\sqrt{2}}} \right]_0^1 = \frac{1}{\sqrt{2}} (\tan^{-1} \sqrt{2})$



(c) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \operatorname{cis} \frac{\pi}{4}$

$\therefore (\operatorname{cis} \frac{\pi}{4})^{2010}$

Every power that is a multiple of 8

ie $(\operatorname{cis} \frac{\pi}{4})^{8n} = 1$

$\therefore (\operatorname{cis} \frac{\pi}{4})^{2008} = 1$

$\therefore (\operatorname{cis} \frac{\pi}{4})^{2010} = (\operatorname{cis} \frac{\pi}{4})^2 = \operatorname{cis} \frac{\pi}{2} = \boxed{i}$

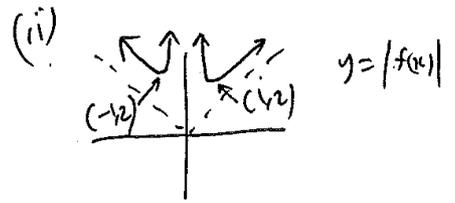
(e) (i) $(x^2+x-6) | 1+12-x = x^2+6$

\therefore Remainder is $12-x$

(d) Now $w = iz$ (90° rotation)

$\therefore z^2 + w^2 = z^2 + i^2 z^2 = z^2 - z^2 = 0$

(e) (i) $z^3 = 1$ or $(z-1)(z^2+z+1) = 0$
 $z^3 = \cos(0+2k\pi)$ $z = 1$ or $z = \frac{-1 \pm \sqrt{-3}}{2}$
 $z = \cos \frac{2k\pi}{3}$
 $k=0: z=1$
 $k=1: z = \cos \frac{2\pi}{3}$
 $k=2: z = \cos \frac{4\pi}{3}$
 $\left. \begin{matrix} k=0: z=1 \\ k=1: z = \cos \frac{2\pi}{3} \\ k=2: z = \cos \frac{4\pi}{3} \end{matrix} \right\} z = 1, \frac{-1 \pm i\sqrt{3}}{2}$

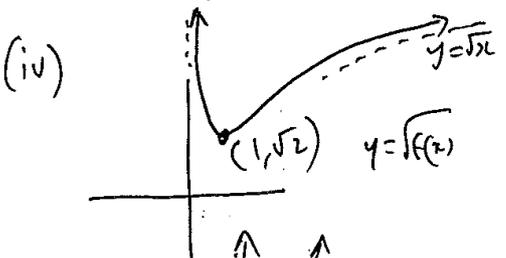
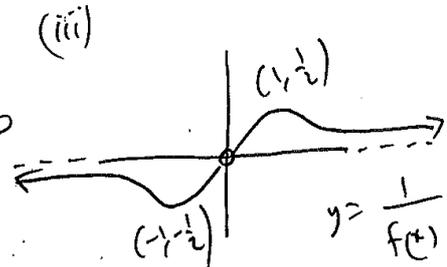


(ii) If λ is unreal, λ satisfies $\lambda^2 + \lambda + 1 = 0$

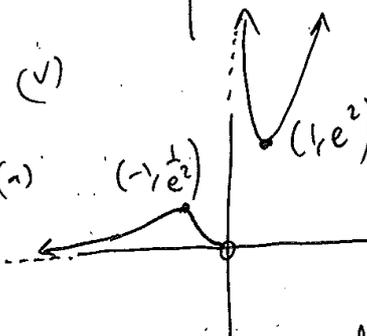
$\therefore \lambda^2 = -\lambda - 1 = -(\lambda + 1)$

(iii) $(1+\lambda)^6 = (-\lambda^2)^6 = \lambda^{12} = (\lambda^3)^4$

Now $\lambda^3 = 1 \therefore (1+\lambda)^6 = 1^4 = 1$ (iv)



(3) (a) $x = \frac{y \pm \sqrt{y^2 - 4}}{2}$ $\therefore y^2 - 4 \geq 0$
 $(y-2)(y+2) \geq 0$
 $\therefore y \geq 2$ or $y \leq -2$
 ie $|y| \geq 2$



(b) (i) $y = x + \frac{1}{x}$
 $xy = x^2 + 1$
 $x^2 - xy + 1 = 0$ ie same graph as in (a)

Thus $y = \pm 2$ are turning points

If $y = 2, 2 = x + \frac{1}{x}$

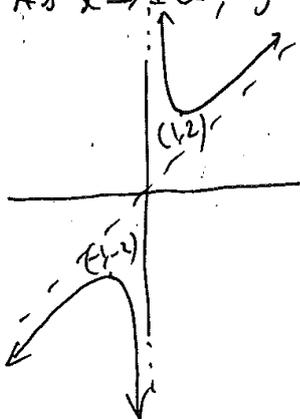
\therefore Min TP $(1, 2)$

If $y = -2, -2 = x + \frac{1}{x}$

\therefore Max TP $(-1, -2)$

Asymptote $x = 0$

As $x \rightarrow \pm \infty, y \rightarrow \pm x \therefore$ Asymptote $y = x$ (ii)



$x^2 - 2x + 1 = 0$
 $(x-1)^2 = 0$
 $x = 1$

$x^2 + 2x + 1 = 0$
 $(x+1)^2 = 0$
 $x = -1$

(c) (i) $\frac{dT}{dt} = kAe^{-kt}$

Now $Ae^{-kt} = 10000 - T$

$\therefore \frac{dT}{dt} = k(10000 - T)$

(ii) $2500 = 10000 - A$
 $\therefore A = 7500$

$5000 = 10000 - 7500e^{-10k}$

$e^{-10k} = \frac{2}{3}$

$\log_e \frac{2}{3} = -10k \therefore k = -\frac{1}{10} \ln \frac{2}{3}$
 $= \frac{\ln \frac{3}{2}}{10}$

\therefore solve $7500 = 10000 - 7500e^{-\frac{\ln \frac{3}{2}}{10} t}$

$\frac{1}{3} = e^{-\frac{\ln \frac{3}{2}}{10} t}$

$t = \frac{-10 \ln \frac{3}{2}}{\ln \frac{3}{2}}$

$\log_e \frac{1}{3} = \frac{\ln \frac{3}{2}}{10} t$

$\therefore 27$ yrs from start

④ (a) (i) Let $f(x) = (x-k)^2 g(x)$
 where $g(k) \neq 0$
 $\therefore f'(x) = 2(x-k)g(x) + g'(x)(x-k)^2$
 $= (x-k) [2g(x) + g'(x)(x-k)]$
 $\therefore f'(k) = 0$

(ii) From (i), $f(1) = 0$ and $f'(1) = 0$
 $\therefore a+b+1=0$ and $7a+6b=0$
 $b = -a-1 \rightarrow 7a+6(-a-1)=0$
 $a=6$
 $b=-7$

(iii) $E'(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$ } let $E(x) = 0$
 ie x is a root
 $\therefore E(x) = E'(x) + \frac{x^4}{24}$

Now if $E(x) = E'(x)$ then x is a double root
 For this to be true, $\frac{x^4}{24} = 0$ ie $x=0$

But if $x=0$, $E(x) = 1$ which is a contradiction
 $\therefore E(x) = 0$ has no double root.

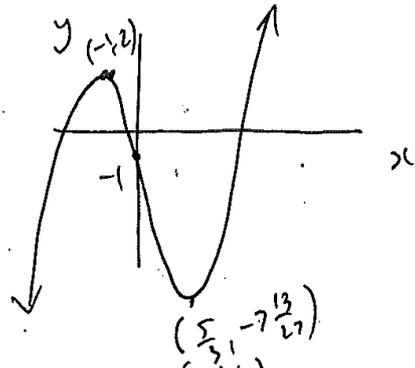
(b) (i) let $y = x^2$
 $\therefore y\sqrt{y} + 7y + 13 = 0$
 $y\sqrt{y} = -7y - 13$
 $y^3 = 49y^2 + 182y + 169$
 $\therefore x^3 - 49x^2 - 182x - 169 = 0$

(ii) Sum of roots of $\dots = 49$

(c) (i) $p(0) = -1$
 $p'(x) = 3x^2 - 2x - 5 = 0$ stat pts
 $(3x-5)(x+1) = 0$

$x = \frac{5}{3}$ or -1
 Stat pts: $(\frac{5}{3}, -\frac{7\frac{13}{27}}{27})$, $(-1, 2)$

$p''(x) = 6x - 2$
 $p''(\frac{5}{3}) = 8 > 0 \therefore$ Min TP $(\frac{5}{3}, -\frac{7\frac{13}{27}}{27})$
 $p''(-1) = -8 < 0 \therefore$ Max TP $(-1, 2)$



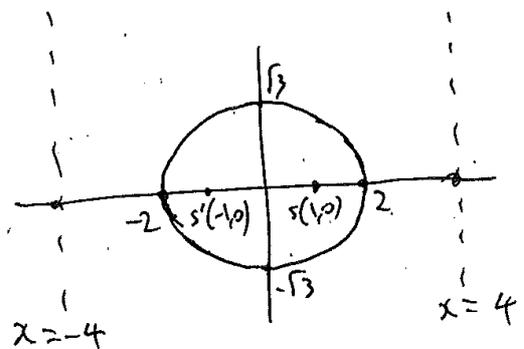
(ii) Polynomial with turning points of opposite sign of y-values must have 3 distinct roots. Since product of roots is 1, they're either all +ve or only one is +ve. Since x-value of one T-P is -ve, must be at least one -ve & thus 2 must be -ve.

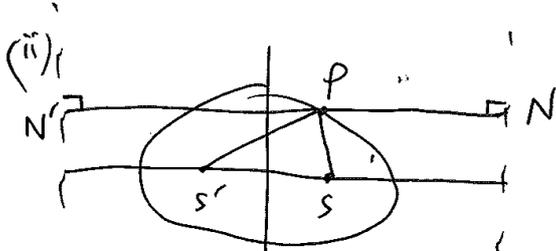
(iii) $x_1 = 0 - \frac{p(0)}{p'(0)}$

$= 0 - \frac{-1}{-5} = -0.2$

(iv) Any $x > \frac{5}{3}$ will converge on the positive root since any tangent will generate x-value $>$ the positive root due to +ve concavity

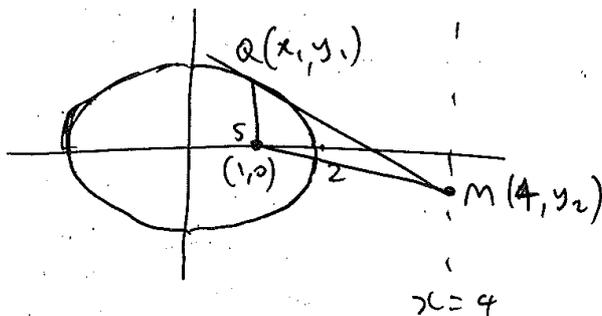
⑤ (a) (i) $a=2$, $b=\sqrt{3}$
 $b^2 = a^2(1-e^2) \therefore 3 = 4(1-e^2)$
 $e^2 = \frac{1}{4} \therefore e = \frac{1}{2}$
 \therefore Foci: $(\pm 1, 0)$
 Directrices: $x = \pm 4$





(ii) The focus-directrix defn says that $PS + PS' = ePN + ePN'$
 $= e(PN + PN') = e(2a) = 2a$
 $\therefore PS + PS' = 2(2) = 4$ units

(iii) The foci in q_n is clearly the positive foci since if QM was (say) the horizontal tangent, $\angle QS'M$ would be acute. Thus the diagram:



Gradient QM : $\frac{x}{2} + \frac{2y}{3} \frac{dy}{dx} = 0$
 $\frac{dy}{dx} = \frac{-3x}{4y} \therefore M_{QM} = \frac{-3x_1}{4y_1}$

Eqn QM : $y - y_1 = \frac{-3x_1}{4y_1}(x - x_1)$
 $4yy_1 - 4y_1^2 = -3xx_1 + 3x_1^2$
 $3x_1^2 + 4y_1^2 = 3xx_1 + 4yy_1$
 $12 = 3xx_1 + 4yy_1$
 $\therefore 1 = \frac{xx_1}{4} + \frac{yy_1}{3}$

Coords M : when $x=4$, $1 = x_1 + \frac{y_2 y_1}{3}$
 $3 = 3x_1 + y_2 y_1$
 $\therefore y_2 = \frac{3(1-x_1)}{y_1} \therefore M(4, \frac{3(1-x_1)}{y_1})$

$$M_{QS} = \frac{y_1}{x_1 - 1}$$

$$M_{MS} = \frac{3(1-x_1)}{y_1} = \frac{1-x_1}{y_1}$$

$$\text{Now } M_{QS} \times M_{MS} = \frac{y_1}{x_1 - 1} \times \frac{1-x_1}{y_1} = -1$$

$\therefore QS \perp MS$ & QM subtends a right-angle at the two foci

(b) (i) x_1 and x_2 are solutions to simultaneous eqns involving L & H.

$$\therefore ax + \frac{bc^2}{x} = 1$$

$$ax^2 - x + bc^2 = 0 \quad \leftarrow \text{has roots } x_1, x_2$$

(ii) Since C is midpt AB ,

$$\frac{x_1 + x_2}{2} = x_0$$

$$\text{But we know that } x_1 + x_2 = \frac{1}{a}$$

$$\therefore \frac{1}{a} = 2x_0$$

$$\therefore \frac{1}{a} = 2x_0 \quad a = \frac{1}{2x_0}$$

Repeating all the steps above for y gives us $b = \frac{1}{2y_0}$

$$\therefore \text{Line } L: \frac{x}{2x_0} + \frac{y}{2y_0} = 1$$

$$\text{or } \frac{x}{x_0} + \frac{y}{y_0} = 2$$

(iii) Centre of rectangular hyperbola is $(0,0)$

\therefore Eqn of line M is $y = kx$

If this line bisects the chords (line L) let it pass through the midpt of the ends of the chords, i.e. through (x_0, y_0)

$$\therefore y_0 = kx_0 \quad \therefore k = \frac{y_0}{x_0} \quad \therefore \text{gradient of line } M \text{ is } \frac{y_0}{x_0}$$

Now gradient of line L : $\frac{y}{y_0} = \frac{-x}{x_0} + 2$

$$y = \frac{-y_0}{x_0}x + 2y_0 \quad \text{which has gradient } \frac{-y_0}{x_0}$$

$\therefore M$ and the chords have the same gradient except one is the negative of the other

(6) (a) (i) $I_n = [e^x x^n]_0^1 - \int_0^1 e^x n x^{n-1} dx$

$I_n = (e - 0) - n \int_0^1 x^{n-1} e^x dx$

$I_n = e - n I_{n-1}$

(ii) $I_2 = e - 2I_1$

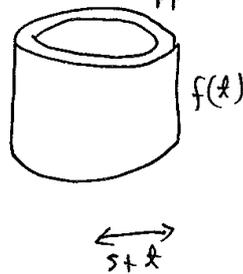
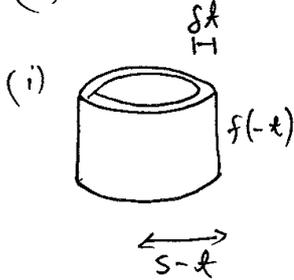
$I_1 = e - I_0$

$I_0 = \int_0^1 e^x dx = [e^x]_0^1 = e - 1$

$\therefore I_1 = e - (e - 1) = 1$

$\therefore I_2 = e - 2(1) = e - 2$

(b) Remember a, s, t are all positive quantities.



$\delta V = 2\pi(s-t)\delta t f(-t)$

$= 2\pi f(-t)(s-t)\delta t$

$\delta V = 2\pi(st+t)\delta t f(t)$

$= 2\pi f(t)(st+t)\delta t$

(ii) $V = \sum_{t=0}^a 2\pi f(-t)(s-t)\delta t + \sum_{t=0}^a 2\pi f(t)(st+t)\delta t$

$= 2\pi \int_0^a f(-t)(s-t) dt + 2\pi \int_0^a f(t)(st+t) dt$

$= 2\pi \int_0^a f(-t)(s-t) + f(t)(st+t) dt$

Now $f(-t) = f(t)$ since $f(x)$ even function

$\therefore = 2\pi \int_0^a f(t)(s-t) + f(t)(st+t) dt$

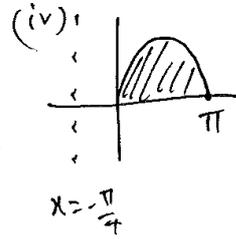
$= 2\pi \int_0^a f(t)[s-t+st+t] dt = 4\pi s \int_0^a f(t) dt$

$= 2\pi \int_0^a 2s f(t) dt$

$= 4\pi s \left(\frac{A}{2}\right)$

$= 2\pi s A$

(iii) $V = 2\pi r s A$ where $A = \pi r^2$
 $\therefore V = 2\pi r s (\pi r^2) = 2\pi^2 r^2 s u^3$



s is the distance from $-\frac{\pi}{4}$ to axis of symmetry of $y = \sin x$ for $0 \leq x \leq \pi$, i.e. distance from $-\frac{\pi}{4}$ to $\frac{\pi}{2}$

$\therefore s = \frac{3\pi}{4}$

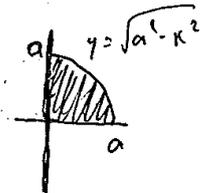
Now $A = \int_0^\pi \sin x dx$

$= [-\cos x]_0^\pi$

$= 1 - (-1) = 2$

$\therefore V = 2\pi \left(\frac{3\pi}{4}\right)(2) = 3\pi^2 u^3$

(c)(i)



$\int_0^a \sqrt{a^2 - x^2} dx$ is the area of a quadrant, radius a

$\therefore = \frac{\pi a^2}{4}$

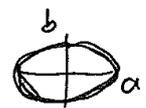
For the ellipse:

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right)$

$y^2 = \frac{b^2}{a^2} (a^2 - x^2)$

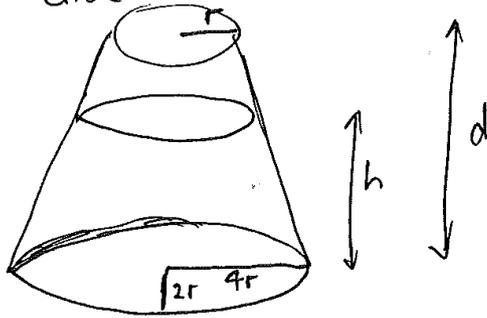
$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$



Area enclosed $= 4 \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx$

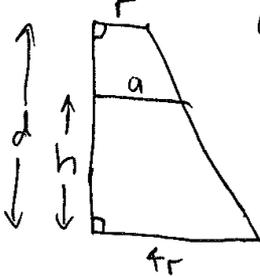
$= 4 \frac{b}{a} \frac{\pi a^2}{4} = \pi ab u^2$

(ii) Circle



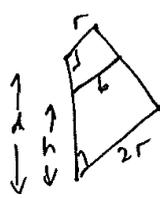
Ellipse

Major axis:



$$a = 4r - \frac{h}{d}(4r - r) \\ = r\left(4 - \frac{3h}{d}\right)$$

Minor axis:



$$b = 2r - \frac{h}{d}(2r - r) \\ = r\left(2 - \frac{h}{d}\right)$$

$$\begin{aligned} \therefore \text{Area} &= \pi ab = \pi r^2 \left(4 - \frac{3h}{d}\right) \left(2 - \frac{h}{d}\right) \\ &= \pi r^2 \left(\frac{4d-3h}{d}\right) \left(\frac{2d-h}{d}\right) \\ &= \frac{\pi r^2}{d^2} (8d^2 - 4dh - 6dh + 3h^2) \\ &= \frac{\pi r^2}{d^2} (3h^2 - 10hd + 8d^2) \end{aligned}$$

$$(iii) \Delta V = A \Delta h$$

$$V = \sum_{h=0}^d A \Delta h$$

$$V = \frac{\pi r^2}{d^2} \int_0^d (3h^2 - 10hd + 8d^2) dh$$

$$V = \frac{\pi r^2}{d^2} \left[h^3 - 5h^2d + 8d^2h \right]_0^d \\ = \frac{\pi r^2}{d^2} [d^3 - 5d^3 + 8d^3] = 4\pi r^2 d$$

⑦ (a) (i) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. For $x > 0$, $\frac{dy}{dx} > 0$
 \therefore Curve increasing for $x > 0$

(ii) Area of rectangles \geq Area under curve

$$\therefore 1(\sqrt{1}) + 1(\sqrt{2}) + 1(\sqrt{3}) + \dots + 1(\sqrt{n-1}) + 1(\sqrt{n}) \\ \geq \int_0^n \sqrt{x} dx$$

$$\therefore \sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n-1} + \sqrt{n} > \int_0^n \sqrt{x} dx$$

(iii) LHS $\geq \left[\frac{2}{3}n^{3/2}\right]_0^n$

$$\text{LHS} \geq \frac{2}{3}n^{3/2}$$

$$\text{LHS} \geq \frac{2}{3}n^{3/2}$$

$$\text{LHS} \geq \frac{2}{3}n\sqrt{n}$$

(iv) since $k > 0$, every quantity is +ve, then squaring both sides does not impact inequality sign

[ie $-4 < 3$ but $(-4)^2 < 3^2$ but in this case not an issue]

$$\therefore \text{PROVE: } (4k+3)^2 k < (4k+1)^2 (k+1)$$

$$\text{LHS} = (16k^2 + 24k + 9)k \\ = 16k^3 + 24k^2 + 9k$$

$$\text{RHS} = (16k^2 + 8k + 1)(k+1) \\ = 16k^3 + 16k^2 + 8k^2 + 8k + k + 1 \\ = 16k^3 + 24k^2 + 9k + 1$$

$$\therefore \text{LHS} < \text{RHS}$$

$$\therefore (4k+3)\sqrt{k} < (4k+1)\sqrt{k+1} \text{ for } k > 0$$

(v) let $n=1$,

$$\text{LHS} = \sqrt{1} = 1 \quad \text{RHS} = \frac{4(1)+3}{6} \sqrt{1} \\ = \frac{7}{6} \times 1 = \frac{7}{6} > 1$$

\therefore True for $n=1$

Let k be an integer such that $\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} \leq \frac{4k+3}{6} \sqrt{k}$
 Prove that $k+1$ is an integer such that $\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \leq \frac{4k+7}{6} \sqrt{k+1}$

Proof: LHS $\leq \frac{(4k+3)\sqrt{k} + \sqrt{k+1}}{6}$ (from assumption)

$$\leq \frac{(4k+3)\sqrt{k} + 6\sqrt{k+1}}{6} \leq \frac{(4k+1)\sqrt{k+1} + 6\sqrt{k+1}}{6} \text{ (from (iv))}$$

$$\leq \frac{\sqrt{k+1}(4k+7)}{6}$$

$$\leq \frac{\sqrt{k+1}(4k+7)}{6}$$

$$\leq \text{RHS}$$

\therefore If true for integer k , is also true for integer $k+1$.
 \therefore If true for $n=1$, is true for $n=2$ & if true for $n=2$, is true for $n=3$
 and so on for all positive integers n

(vi) From (iii) & (v), $\frac{2n\sqrt{n}}{3} \leq \sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$

$$\therefore \frac{20000(100)}{3} \leq \sqrt{1} + \sqrt{2} + \dots + \sqrt{10000} \leq \frac{40003(100)}{6}$$

$$\therefore 666,666.6 \leq \sqrt{1} + \sqrt{2} + \dots + \sqrt{10000} \leq 666,716.6$$

$$\therefore \sqrt{1} + \sqrt{2} + \dots + \sqrt{10000} \approx 666,700 \text{ (nearest hundred)}$$

(b)(i) Solve $x^7 = 1$
 $x = \left\{ \cos\left(\frac{0+2k\pi}{7}\right) \right\}^{1/7}$
 $x = \cos\frac{2k\pi}{7}$. If $k=1$,
 $\therefore q$ is a complex 7th root of unity

Alternatively, to solve $x^7 = 1$,
 $x^7 - 1 = 0 \Rightarrow (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0$
 $x=1$ is the real soln & $q \neq 1$
 $\therefore q$ is a soln to the 2nd bracket,
 ie $1 + q + q^2 + q^3 + q^4 + q^5 + q^6 = 0$

(ii) Since a and b are real, roots are in conjugate pairs. $\therefore \beta = q + q^2 + q^4$
 $\beta = \bar{q} + \bar{q}^2 + \bar{q}^4 = q^6 + q^5 + q^3$
 (since $\bar{q} = q^{-1} = q^6$ etc)

(iii) $\alpha + \beta = -a$
 $\therefore q + q^2 + q^4 + q^6 + q^5 + q^3 = -a$
 $-1 = -a$
 $\therefore \boxed{a=1}$

$\alpha + \beta = 6 \Rightarrow (q + q^2 + q^4)(q^6 + q^5 + q^3) = 6$
 $(q^7 + q^6 + q^4 + q^8 + q^7 + q^5 + q^{10} + q^9 + q^7) = 6$
 $1 + 1 + 1 + q^6 + q^4 + q^5 + q^3 + q^2 = 6$
 [since $q^7 = 1$ and $q^8 = q$ etc]
 $\therefore 3 + -1 = 6 \Rightarrow \boxed{b=2}$

(iv) $x^2 + x + 2 = 0$ has roots
 $x = \frac{-1 \pm i\sqrt{7}}{2}$. Equating imaginary parts,

$\frac{\sqrt{7}}{2} = \text{Imag}\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}\right)$
 or $\text{Imag}\left(\cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} + \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} + \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}\right)$

$\therefore \frac{\sqrt{7}}{2} = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$
 or $\sin\frac{6\pi}{7} + \sin\frac{10\pi}{7} + \sin\frac{12\pi}{7}$
 $\therefore \frac{\sqrt{7}}{2} = \underbrace{\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7}}_{+ve}$ or $\underbrace{\sin\frac{\pi}{7} - \sin\frac{3\pi}{7} - \sin\frac{2\pi}{7}}_{-ve}$

$\therefore \frac{\sqrt{7}}{2} = \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7}$
 $\therefore \sin\frac{\pi}{7} - \sin\frac{2\pi}{7} - \sin\frac{3\pi}{7} = \frac{-\sqrt{7}}{2}$

(3)(a)(i) $\frac{dv}{dx} = -\left(\frac{400+v^2}{40v}\right)$

$\therefore \frac{dx}{dv} = \frac{-40v}{400+v^2} = (-20) \frac{2v}{400+v^2}$

$\therefore x = -20 \ln(400+v^2) + C$

When $x=0, v=20 \therefore 0 = -20 \ln 800 + C$

$C = 20 \ln 800$

$\therefore x = 20 \ln\left(\frac{800}{400+v^2}\right)$

Max height when $v=0 \therefore = 20 \ln 2$ metres

(ii) $\frac{dt}{dv} = \frac{-40}{400+v^2}$

$\therefore t = \frac{-40}{20} \tan^{-1} \frac{v}{20} + C \therefore t = C - 2 \tan^{-1} \frac{v}{20}$

When $v=20, t=0 \therefore 0 = C - 2 \tan^{-1} 1$
 $0 = C - \frac{\pi}{2} \therefore C = \frac{\pi}{2}$

$\therefore t = \frac{\pi}{2} - 2 \tan^{-1} \frac{v}{20}$

Max height when $v=0 \therefore \frac{\pi}{2}$ seconds

(iii) $x = 10 - \frac{1}{40} v^2$

(iv) $v \frac{dv}{dx} = \frac{400-v^2}{40} \therefore \frac{dx}{dv} = \frac{40v}{400-v^2}$

$\frac{dx}{dv} = (-20) \frac{-2v}{400-v^2}$

$\therefore x = -20 \ln(400-v^2) + C$

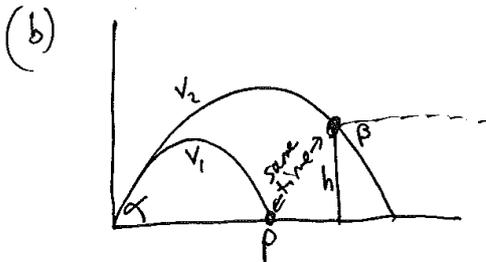
When $x=0, v=0 \therefore 0 = -20 \ln(400-0^2) + C$

$\therefore C = 20 \ln 400$

$\therefore x = 20 \ln\left(\frac{400}{400-v^2}\right)$

At starting point, $x = 20 \ln 2$

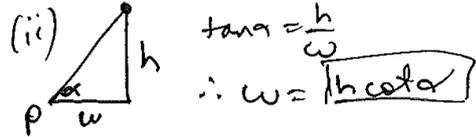
$\therefore 2 = \frac{400}{400-v^2} \therefore 400-v^2 = 200$
 $v^2 = 200$
 $v = 10\sqrt{2}$ m/s



(i) slow stone Fast stone

$x_{v_1} = v_1 t \cos \alpha$ $x_{v_2} = v_2 t \cos \alpha$
 $y_{v_1} = v_1 t \sin \alpha - \frac{gt^2}{2}$ $y_{v_2} = v_2 t \sin \alpha - \frac{gt^2}{2}$

Gradient between 2 stones = $\frac{y_{v_2} - y_{v_1}}{x_{v_2} - x_{v_1}} = \frac{t \sin \alpha (v_2 - v_1)}{t \cos \alpha (v_2 - v_1)}$
 $= \tan \alpha \therefore$ independent of time



(iii) Fast stone: $\tan\left(\frac{\beta}{2}\right) = -\beta$

$\tan\left(\frac{v_2 \sin \alpha - gt}{v_2 \cos \alpha}\right) = -\beta$

$\therefore \tan(-\beta) = \frac{v_2 \sin \alpha - gt}{v_2 \cos \alpha}$

At this moment, slow stone hits ground

$\therefore 0 = v_1 \sin \alpha - \frac{gt^2}{2} = t \left(v_1 \sin \alpha - \frac{gt}{2} \right)$

$\therefore t = \frac{2v_1 \sin \alpha}{g}$
start time

$\therefore \tan(-\beta) = \frac{v_2 \sin \alpha - 2v_1 \sin \alpha}{v_2 \cos \alpha}$

$\tan(-\beta) = \frac{\sin \alpha (v_2 - 2v_1)}{v_2 \cos \alpha}$

$-\tan \beta = \left(\frac{v_2 - 2v_1}{v_2}\right) \tan \alpha$

$\therefore -v_2 \tan \beta = v_2 \tan \alpha - 2v_1 \tan \alpha$

$v_2 (\tan \alpha + \tan \beta) = 2v_1 \tan \alpha$

Now if $2\beta = \alpha, v_2 (\tan 2\beta + \tan \beta) = 2v_1 \tan 2\beta$

$v_2 \left[\frac{2 \tan \beta + \tan \beta}{1 - \tan^2 \beta} \right] = \frac{(2v_1) 2 \tan \beta}{1 - \tan^2 \beta} [x | - \tan^2 \beta]$

$\therefore v_2 (2 \tan \beta + \tan \beta - \tan^3 \beta) = 4v_1 \tan \beta$

Since $\tan \beta \neq 0, \div \tan \beta \therefore v_2 (3 - \tan^2 \beta) = 4v_1$

$3v_2 - v_2 \tan^2 \beta = 4v_1$

$\tan^2 \beta = \frac{3v_2 - 4v_1}{v_2}$ Now since $\tan^2 \beta > 0,$
and $v_2 > 0,$

$3v_2 - 4v_1 > 0$

$\therefore 3v_2 > 4v_1$ or $v_1 < \frac{3}{4} v_2$